ECONTUTORIALS · FOR TEACHERS

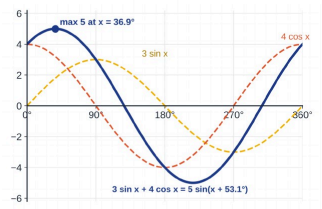
Mathematics Made Simple

TEACHER PACK

20 ready-to-teach chapters for AP Calculus, Cambridge 9709 and IB

SLIDES · LESSON PLANS · TESTS · EXAMS · TEXTBOOK + PRACTICE BOOK INCLUDED

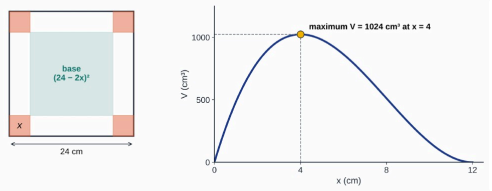
Two waves make one: 3 sin x + 4 cos x



- 3 sin x and 4 cos x have the same period
- Their sum is one wave of amplitude 5
- Max 5 at x = 90° − 53.1° = 36.9°
- The phase shift is α = 53.1° to the left

Mathematics Made Simple · Teacher Pack · Chapter 8: Trigonometry II 16

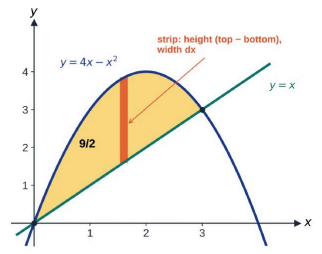
Optimisation: the biggest open box



- Cut squares of side x from a 24 cm square sheet: V = x(24 − 2x)², 0 < x < 12
- V′ = (24 − 2x)(24 − 6x) = 0 at x = 4 (x = 12 gives V = 0)
- V″(4) < 0, so maximum volume V(4) = 1024 cm³

Mathematics Made Simple · Teacher Pack · Chapter 12: Applying Derivatives 16

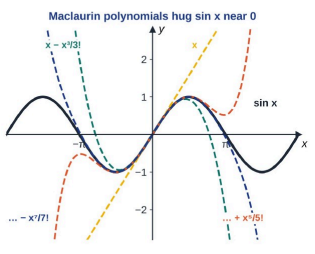
The area between y = 4x − x² and y = x



- Intersections: 4x − x² = x gives x = 0 and x = 3
- Top: 4x − x². Bottom: x
- Area = $\int_0^3 (3x - x^2)\,dx = 27/2 - 9 = 9/2$
- One representative strip shows the method

Mathematics Made Simple · Teacher Pack · Chapter 15: Applications of Integration 5

Taylor polynomials approximate sin x



- sin x = x − x³/3! + x⁵/5! − x⁷/7! + …
- Each extra term follows sin x further from 0
- The series converges to sin x for every real x
- Near 0, sin x ≈ x (small-angle approximation)

Mathematics Made Simple · Teacher Pack · Chapter 20: Parametric, Polar and Infinite Series 18

THE ECONTUTORIALS TEAM

ECONTUTORIALS.COM

CHAPTER 12

Applying Derivatives

12

Two ready-to-teach lessons · Mathematics Made Simple

AP Calculus 4.1–4.7, 5.1–5.11
Cambridge 9709 P1 1.7
IB AA SL 5.2, 5.7–5.9, HL 5.13

EconTutorials

Learning objectives

1. Analyse straight-line motion using velocity and acceleration
2. Use linearisation to estimate values and L’Hôpital’s rule for 0/0 limits
3. Apply the Mean Value Theorem and find absolute extrema with the candidates test
4. Classify stationary points with the first and second derivative tests, and find inflection points
5. Solve optimisation problems and justify that the answer is a maximum or minimum

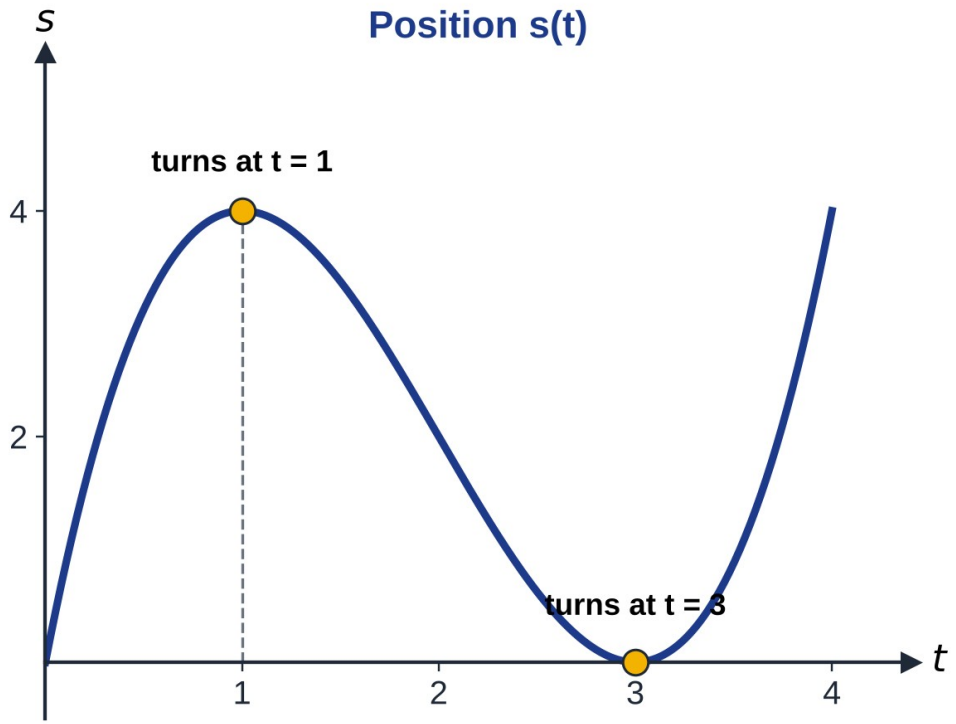
Syllabus alignment

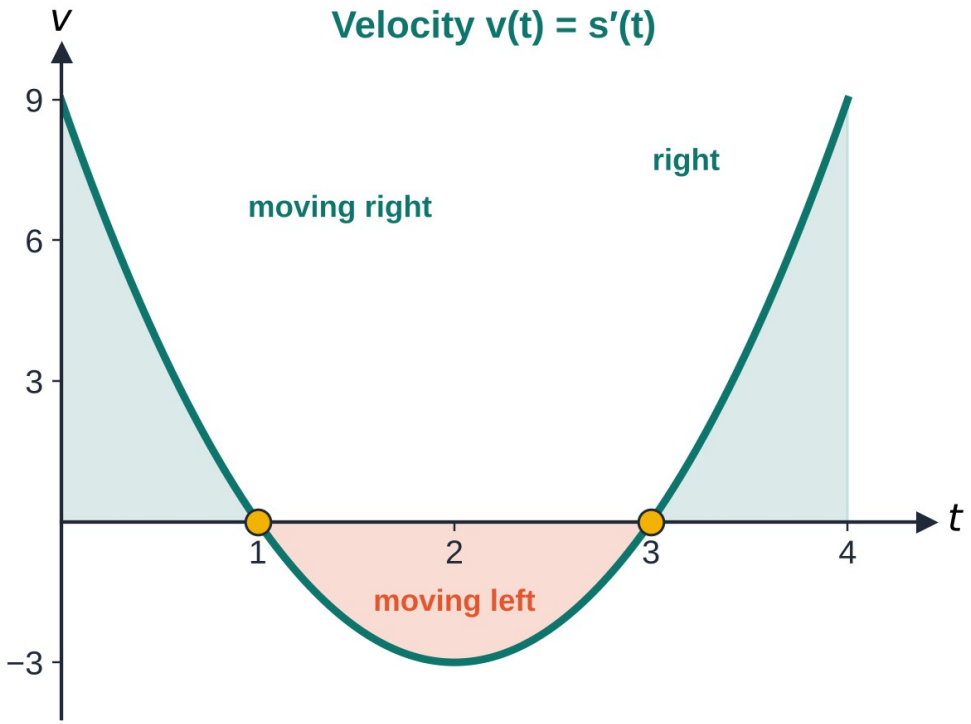
AP Calculus AB/BC
Unit 4: 4.1–4.3, 4.6–4.7; Unit 5: 5.1–5.11

Cambridge 9709
P1 1.7

IB Maths AA
SL 5.2, 5.7–5.9; HL 5.13

Position and velocity: $s = t^3 - 6t^2 + 9t$





- $v = 3t^2 - 12t + 9 = 3(t - 1)(t - 3)$: at rest at $t = 1$ and $t = 3$
- The particle moves right, then left ($1 < t < 3$), then right again
- Turning points of s line up with the zeros of v

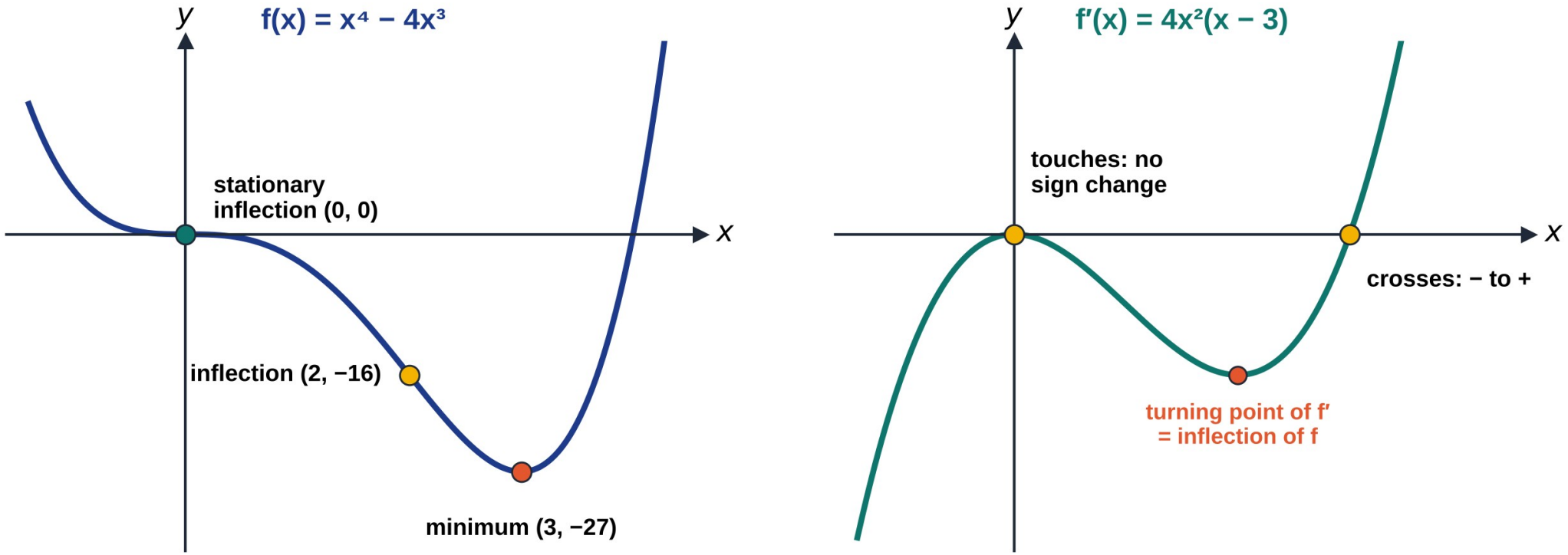
Worked example: analysing motion

A particle moves with $s = t^3 - 6t^2 + 9t$ metres for $0 \le t \le 4$ seconds. When is it speeding up, and how far does it travel?

1. $v = 3(t - 1)(t - 3)$, $a = 6t - 12$. At rest at $t = 1$ and $t = 3$
2. $v < 0$ on $(1, 3)$; $a < 0$ on $(0, 2)$. Same sign on $(1, 2)$ and on $(3, 4)$: speeding up
3. $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(4) = 4$
4. Distance $= 4 + 4 + 4 = 12$ m, but displacement $= s(4) - s(0) = 4$ m

Answer Speeding up for $1 < t < 2$ and $3 < t < 4$. Total distance 12 m; displacement 4 m.

Reading f, f′ and f″: $y = x^4 - 4x^3$



- $f' = 4x^2(x - 3)$: stationary at $x = 0$ (no sign change) and $x = 3$ (minimum, $y = -27$)
- $f'' = 12x(x - 2)$: inflection at $x = 0$ and at $x = 2$
- At $x = 0$ the curve is flat and changes concavity: a stationary inflection

Check for understanding

$f'(x) = (x - 2)^2(x + 1)$. At $x = 2$, f has…

A a local maximum

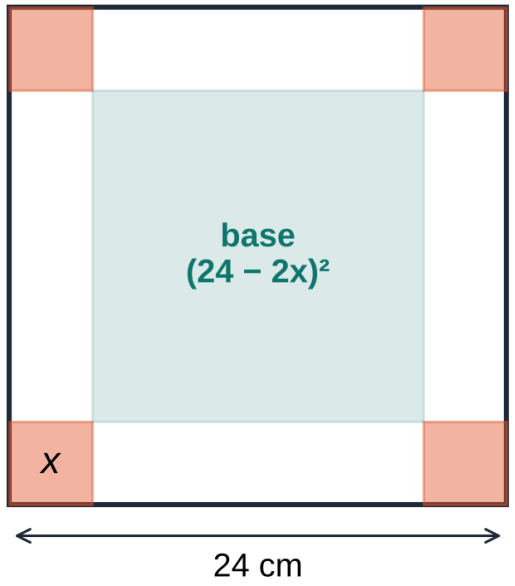
B a local minimum

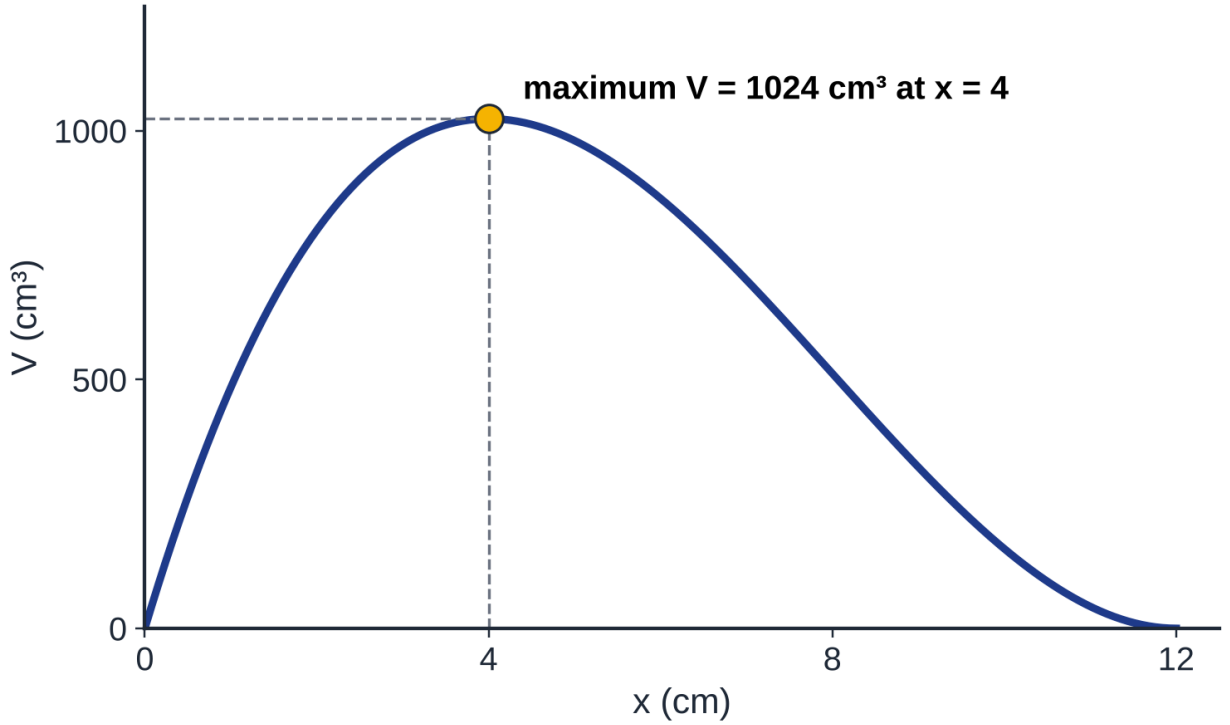
C neither: f' does not change sign

D an absolute maximum

Show answers on mini-whiteboards: A, B, C or D

Optimisation: the biggest open box





- Cut squares of side x from a 24 cm square sheet: $V = x(24 - 2x)^2$, $0 < x < 12$
- $V' = (24 - 2x)(24 - 6x) = 0$ at $x = 4$ ($x = 12$ gives $V = 0$)
- $V''(4) < 0$, so maximum volume $V(4) = 1024\ \text{cm}^3$

Worked example: maximising profit

A firm faces demand $p = 100 - 2q$ and has costs $C = 50 + 10q + q^2$. Find the output that maximises profit, and the maximum profit.

1. Revenue $R = pq = 100q - 2q^2$. Profit $\pi = R - C = -3q^2 + 90q - 50$
2. $d\pi/dq = -6q + 90 = 0$, so $q = 15$
3. $d^2\pi/dq^2 = -6 < 0$, so this is a maximum
4. $\pi(15) = -675 + 1350 - 50 = 625$; price $p = 100 - 30 = 70$

Answer Profit is maximised at $q = 15$ units, price 70, giving maximum profit 625.

Case study: the world’s fastest roller-coaster launch

1. Formula Rossa at Ferrari World, Abu Dhabi, holds the record for the fastest roller-coaster launch
2. It goes from 0 to 240 km/h in 4.9 seconds
3. A hydraulic launcher sends the cars onto a 2.2 km track

13.6 m/s²

average acceleration during the launch (about 1.4 g)

Discuss Use the Mean Value Theorem to explain why, at some instant during the launch, the acceleration was exactly 13.6 m/s². What must you assume about v(t)?

Source: Guinness World Records, “Highest speed achieved by a rollercoaster launch” (Formula Rossa, Ferrari World Abu Dhabi), guinnessworldrecords.com/world-records/780255-highest-speed-achieved-by-a-rollercoaster-launch

CHAPTER 12 · LESSON PLAN

Applying Derivatives

Two 60-minute lessons · Slides: Ch12-Applying-Derivatives-Slides.pptx · Worksheet, test and exit tickets in this pack

AP	AP Calculus AB and BC Unit 4: topics 4.1–4.3 (rates in context and straight-line motion) and 4.6–4.7 (linearisation, L'Hôpital's rule); Unit 5: topics 5.1–5.11 (Mean Value and Extreme Value Theorems, derivative tests, concavity, curve sketching, optimisation)
Cambridge	Cambridge International AS & A Level Mathematics 9709, Pure 1 topic 1.7 (rates of change, stationary points, maxima and minima, and practical optimisation)
IB	IB Mathematics: Analysis and Approaches SL topics 5.2 (increasing and decreasing functions), 5.7 (the second derivative), 5.8 (stationary points, inflection and optimisation) and 5.9 (kinematics); HL topic 5.13 (L'Hôpital's rule)
Prior learning	All the differentiation rules of Chapter 10, including the chain rule and derivatives of e^x, ln x and trigonometric functions; sketching polynomials (Chapter 3)
Resources	Slide deck, student worksheet (one per student), mini-whiteboards, calculators, exit tickets

Learning objectives

1. Analyse straight-line motion using velocity and acceleration.
2. Use linearisation to estimate values and L'Hôpital's rule for 0/0 limits.
3. Apply the Mean Value Theorem and find absolute extrema with the candidates test.
4. Classify stationary points with the first and second derivative tests, and find inflection points.
5. Solve optimisation problems and justify that the answer is a maximum or minimum.

Key vocabulary

Term	Meaning
Velocity and acceleration	$v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$. Speed is $\lvert v(t) \rvert$.
Linearisation	$L(x) = f(a) + f'(a)(x - a)$: the tangent line at a, used to estimate f(x) for x near a.
L'Hôpital's rule	If $f(x)/g(x) \to 0/0$ or ∞/∞, then $\lim f/g = \lim f'/g'$, provided the second limit exists.
Mean Value Theorem	If f is continuous on [a, b] and differentiable on (a, b), some c in (a, b) has $f'(c) = (f(b) - f(a))/(b - a)$.
Critical point	A point in the domain where $f'(x) = 0$ or $f'(x)$ does not exist.
Stationary point	A point where $f'(x) = 0$: a local maximum, a local minimum or a stationary point of inflection.
Point of inflection	A point where the curve changes concavity, so f'' changes sign.
Absolute (global) extremum	The largest or smallest value of f on the whole interval; on [a, b] it occurs at a critical point or an endpoint.

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CHAPTER 12 · STUDENT WORKSHEET

Applying Derivatives

Show your working and justify every maximum, minimum or inflection. Give units where there is a context.

Name: Class: _____ Date: _____

_____

_

Formulas v = ds/dt a = dv/dt L(x) = f(a) + f′(a)(x − a) lim f/g = lim f′/g′ (0/0 or ∞/∞) MVT: f′(c) = (f(b) − f(a))/(b − a)

Task 1: Motion [8 marks]

A particle moves along a line with position $s = t^3 - 9t^2 + 24t$ metres, for $0 \le t \le 5$ seconds.

(a) Find v(t) and a(t), and the times when the particle is at rest.

(b) Find the intervals when the particle is speeding up.

(c) Find the total distance travelled in the 5 seconds, and the displacement.

Task 2: Linearisation and L'Hôpital [7 marks]

(a) Use the linearisation of √x at x = 25 to estimate √26. Is this an overestimate or an underestimate? Explain.

(b) Find the limit of sin(3x)/(2x) as $x \to 0$.

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CHAPTER 12 · CHAPTER TEST

Applying Derivatives

Time: 50 minutes · Total: 50 marks · Calculator use is shown on each question

Name: Class: _____ Date: _____

_____

_

Section A: Multiple choice [10 marks]

Circle one answer for each question.

1. $s(t) = t^2 - 4t$. At $t = 1$ the particle is

A speeding up **B** slowing down

C at rest **D** moving with constant speed

2. The linearisation of $f(x) = \ln x$ at $x = 1$ is

A $L(x) = x - 1$ **B** $L(x) = x$

C $L(x) = 1/x$ **D** $L(x) = x + 1$

3. The limit of $(e^x - 1 - x)/x^2$ as $x \to 0$ is

A 0 **B** 1

C 1/2 **D** 2

4. Which function does NOT satisfy the MVT conditions on $[-1, 1]$?

A x^2 **B** x^3

C $\sin x$ **D** $|x|$

5. $f'(x)$ changes from negative to positive at $x = 3$. Then f has

A a local maximum at 3 **B** a local minimum at 3

C an inflection at 3 **D** an absolute maximum at 3

6. $f''(x) = 6x - 12$. The curve is concave down for

A $x > 2$ **B** $x < -2$

C $x < 2$ **D** all x

7. The absolute maximum of $f(x) = x^2$ on $[-1, 3]$ is

A 9 **B** 1

C 0 **D** 3

8. $y = x^4$ has $f'(0) = f''(0) = 0$. At $x = 0$ the curve has

A a maximum **B** an inflection

C no stationary point **D** a minimum

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CHAPTER 12 · CHAPTER TEST MARK SCHEME

Applying Derivatives

Total 50 marks. Award marks for correct mathematics even where wording or method differs (follow-through marks where shown).

Section A: Multiple choice

Q	Answer	Reason
1	B	$v = 2t - 4 = -2$ and $a = 2$ have opposite signs.
2	A	$f(1) = 0$, $f'(1) = 1$: $L(x) = 0 + 1(x - 1)$.
3	C	L'Hôpital twice: $(e^x - 1)/(2x)$, then $e^x/2 \to 1/2$.
4	D	$\lvert x \rvert$ is not differentiable at 0.
5	B	First derivative test: − to + is a minimum.
6	C	$f'' < 0$ when $6x - 12 < 0$.
7	A	Candidates: $f(-1) = 1$, $f(0) = 0$, $f(3) = 9$.
8	D	$f' = 4x^3$ changes from − to +: minimum (second derivative test inconclusive).
9	C	$S = x + 36/x$, $S' = 1 - 36/x^2 = 0$ at $x = 6$: sum 12.
10	B	$a = 6t - 6 = 6$ at $t = 2$.

Question 11: A moving particle (no calculator) [10]

(a) $v = 6t^2 - 30t + 36$ B1; $a = 12t - 30$ B1.

(b) $v = 6(t - 2)(t - 3) = 0$ M1; $t = 2$ and $t = 3$ A1; $a(2) = -6$ m/s², $a(3) = 6$ m/s² A1.

(c) $s(0) = 0$, $s(2) = 28$, $s(3) = 27$, $s(4) = 32$ M1 (splitting at the turning points); $28 + 1 + 5$ M1; $= 34$ m A1.

(d) $v(2.8) = -24/25 < 0$ and $a(2.8) = 18/5 > 0$ B1; opposite signs, so slowing down B1.

Question 12: The cheapest tin (calculator allowed) [10]

(a) $V = \pi r^2 h = 1000$ B1; $h = 1000/(\pi r^2)$ M1; $A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2000/r$ A1 (answer given).

(b) $dA/dr = 4\pi r - 2000/r^2$ M1; $= 0$ gives $r^3 = 500/\pi$ M1; $r \approx 5.42$ cm A1.

(c) $d^2A/dr^2 = 4\pi + 4000/r^3$ M1; > 0 for $r > 0$, so a minimum A1.

(d) $h = 1000/(\pi r^2) = 1000r/(\pi r^3) = 1000r/500 = 2r$ B1; $A = 6\pi r^2 \approx 553.6$ cm² B1.

Question 13: Extended problem — analysing $f(x) = x^2e^{-x}$ [20]

Marking points (award each where seen; allow follow-through where indicated):

- (a) Product rule: $2xe^{-x} - x^2e^{-x}$ M1 A1; $= xe^{-x}(2 - x)$ (shown); stationary points (0, 0) B1 and $(2, 4e^{-2})$ B1.
- (b) Differentiate again to obtain $e^{-x}(x^2 - 4x + 2)$ M1 A1 (shown); $f''(0) = 2 > 0$: minimum; $f''(2) = -2e^{-2} < 0$: maximum A1.
- (c) $x^2 - 4x + 2 = 0$ M1; $x = 2 \pm \sqrt{2}$ A1; $e^{-x} > 0$ so f'' changes sign at both: both are inflections A1.

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